Analysis of Damaged Structure

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A FAIL-SAFE or survivability analysis of a complex structure would require the determination of the strength of the structure after some structural damage (e.g., projectile penetration) has occurred. One procedure would be to reanalyze the damaged structure as if it were a brand new structure. This can be quite time consuming, even when a digital computer is available, since each possible combination of damage requires a rephrasing of the structural problem. It also fails to give the engineer a "physical feel" of the resultant structural behavior. Another procedure, which is recommended in this Note, is to utilize a method that can be described as one-step generalized "Hardy-Cross" solution. Most engineers are familiar with the Hardy-Cross method, which presents a visual picture and physical meaning to each step employed to obtain a "relaxation" solution. The proposed method requires a minimum amount of computation since it utilizes the results of the analysis for the undamaged structure that is available.

The method is derived mathematically by performing the Hardy-Cross distribution a number of times and then going to the limit rather than stopping at some acceptable value. The logic is discussed below.

1) The solution for the internal loads in the undamaged structure is first obtained as

$$F_0(M) = f(M,H)\ P(H) \tag{1}$$

where $F_0(M)$ = internal loads at the M degrees of freedom; $P(H)$ = external loads (including zero) at the H degrees of freedom; and $f(M,H)$ = transformation matrix that converts the external loads to the internal loads $\{$e.g., $f(M,H) = k(M,H)\ [K(H,\ H)]^{-1}\}$. $k(M,H)$ is the stiffness (force) of a structural element at the M degrees of freedom for unit displacements at H, and $K(H,H) = \Sigma_{M=H}k(M,H)$. H is a subset of M for which the displacements are nonzero.

2) Hypothetical jacks are applied to the structure so that the joints cannot move from their deflected position;

$$\Delta_0(M) = \delta(M,H)\ P(H) \tag{2}$$

where $\Delta_0(M)$ = deflection of the M degrees of freedom; and $\delta(M,H)$ = influence coefficients, flexibility of M due to unit load at H $\{$e.g., $\delta(M,H) = [K(H,H)]^{-1} + 0\ (M - H,\ H)\}$.

3) Select the K degrees of freedom that will be damaged so that they are incapable of carrying any internal loads. As an example, shooting away a root beam will destroy its shear and moment capacity and result in a redistribution of the internal loads. It should be noted that the proposed procedure will result in the solution (loads and deformations) for a "linear" structure with a known internal loading (e.g., zero, yield, or instability) at the damaged degrees of freedom.

4) Remove the jacks at the K degrees of freedom and then mathematically disassociate these degrees of freedom from the structure. This is effected by applying the negative of the computed internal loads at these points [i.e., $-F_0(K)$] and reacting them with the remaining clamped structure. Because the structure is clamped by the hypothetical jacks, the load is felt only by a few surrounding jacks, i.e.,

$$\lambda_1(J) = B(J,K)[-F_0(K)] \tag{3}$$

where $\lambda_1(J)$ is the load in the J jacks, which are a subset of the H jacks and bound the K degrees of freedom, and $B(J,K)$ are the loads felt in the J jack for a unit load at the K degrees of freedom. A few typical B matrices are shown in Fig. 1. The released ends of the K degrees of freedom deflect an additional amount with respect to the structure, i.e.,

$$\bar{\Delta}_0(K) = \bar{\delta}(K,K)[-F_0(K)] \tag{4}$$

where $\bar{\delta}(K,K)$ is the influence coefficients for the cut ends with respect to the datum represented by the J jacks (same as the structure).

5) Compatibility jacks are then added between each end of the K degrees of freedom which keep the points $\bar{\Delta}_0(K)$ apart and then all the other jacks are removed. The load in the J jacks is then applied to this structure. The structural behavior is exactly the same as the original undamaged structure, since equilibrium and compatibility is satisfied at all degrees of freedom [each side of the K degrees of freedom do not move closer or further apart during the $\lambda(J)$ loading]. Thus,

$$F_1(M) = f(M,J)\lambda_1(J) = f(M,J)B(J,K)[-F_0(K)] \tag{5}$$

and

$$\Delta_1(M) = \delta(M,J)\lambda_1(J) = \delta(M,J)B(J,K)[-F_0(K)] \tag{6}$$

represent the incremental internal loads and deflections of the structure that must be added to $F_0(M)$, $\bar{\Delta}_0(M)$, and $\bar{\Delta}_0(K)$ to obtain an intermediate solution to the damaged structure.

6) The internal loads that exist at the K degrees of freedom at this stage of the analysis are

$$F_0(K) - F_0(K) + f(K,J)\lambda_1(J) = f(K,J)\lambda_1\ (J) \tag{7}$$

Since it is not zero or sufficiently small to accept the solution we must iterate steps 2–5 until we are satisfied with the solution. This is similar to the Hardy-Cross approach, where the structure is fixed by jacks at all degrees of freedom (step 2); load is applied to structure and reacted at the jacks (step 4); selected jacks are released (step 3); and the un-

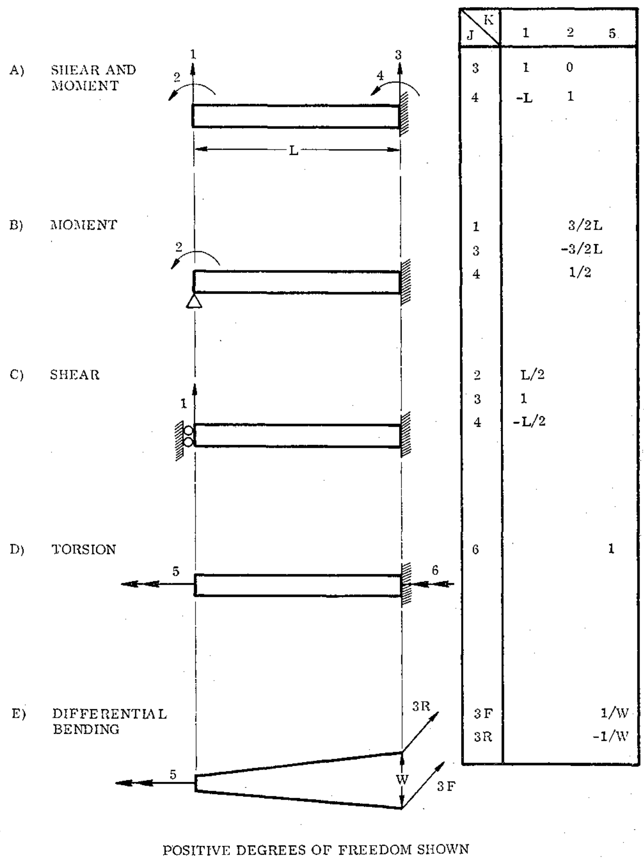

Fig. 1 Typical B matrices.

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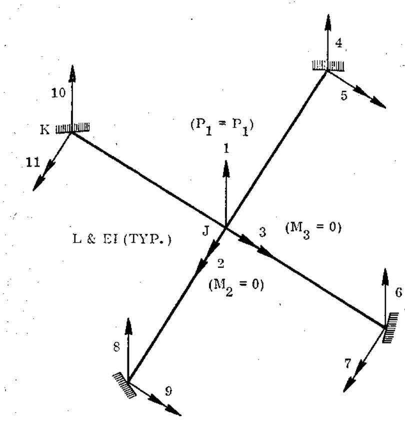

Fig. 2 Structural example.

balance load redistributed to the resultant structure; at which time, the procedure is iterated until satisfactory unbalances (load in hypothetical jacks) are sufficiently small.

A repeat of steps 2–6 results in

$$\lambda_2(J) = B(J,K)[-F_1(K)] = -B(J,K)f(K,J)\lambda_1(J) \quad (8a)$$

and

$$\lambda_n(J) = B(J,K)[-F_{n-1}(K)] = -B(J,K)\,f(K,J)\lambda_{n-1}(J) = [-B(J,K)f(K,J)]^{n-1}\lambda_1(J) \quad (8b)$$

Similarly

$$\bar{\Delta}_n(K) = \bar{\delta}(K,K)f(K,J)\lambda_n(J) \quad (9a)$$

and

$$\Delta_n(M) = \delta(M,J)\lambda_n(J) \quad (9b)$$

Rather than stopping at n iterations we can very easily sum up the increments to obtain the limiting value (solution). Thus,

$$R(J) = \sum_{n=1}^{\infty} \lambda_n(J) = \sum_{i=0}^{\infty} [-B(J,K)f(K,J)]^i\lambda_1(J) \quad (10)$$

$$\therefore R(J) = [I(J,J) + B(J,K)f(K,J)]^{-1}B(J,K)[-F_0(K)]$$

This will require inversion of a small $J \times J$ matrix (usually two by two, see Fig. 1) which is much smaller than the matrix inversion required to solve the original structure. The final results are

$$\bar{F}(M) = \Sigma F_i(M) = f(M,H)P(H) + f(M,J)R(J) = \text{internal loads}, M \neq K \quad (11a)$$

and

$$\bar{F}(K) = \Sigma F_i(K) = F_0(K) + -F_0(K) = 0 \quad (11b)$$

$$D(M) = \Sigma\Delta(M) = \delta(M,H)P(H) + \delta(M,J)R(J) = \text{displacement of structure} \quad (12a)$$

$$\bar{D}(K) = \Sigma\Delta(K) + \Sigma\bar{\Delta}(K) = D(K) + \bar{\delta}(K,K)f(K,J)R(J) = \text{displacement of damaged area} \quad (12b)$$

The value $-F_0(K)$ of Eq. (10) represents the change in the internal loads at the K degrees of freedom which produces the resulting loads and deflections defined by Eqs. (11) and (12). This value need not be equal and opposite to $F_0(K)$ but could be any value [e.g., $F(K) = F_{\text{yield}}(K) - F_0(K)$ or even a unit loading] chosen by the analyst.

The only restriction is that the behavior of the resulting structure is correctly defined by Eq. (1) (remains linear). Although this would not restrict the number of destroyed elements, it would limit a given solution to the yielding or instability of only one degree of freedom at a time. Multiple nonlinearities can be analyzed by obtaining the subsequent, but different, behavior [$\bar{f}(M,H)$ as indicated below] after each successive nonlinearity.

To obtain the unit solution applicable to the damaged structure it is only necessary to utilize a $F_0(K)$ corresponding to unit loadings, i.e.,

$$P(H) \rightarrow I(H,H) \quad (13a)$$

$$F_0(K) \rightarrow f(K,H) \quad (13b)$$

$$\bar{F}(M) \rightarrow \bar{f}(M,H) \quad (13c)$$

Substituting in Eqs. (10) and (11) we obtain

$$\bar{f}(M,H) = f(M,H) - f(M,J)[I(J,J) + B(J,K)f(K,J)]^{-1}B(J,K)f(K,H) \quad (14)$$

where $\bar{f}(M,H)$ is the matrix that transforms the external loads to internal loads for the damaged structure. Again it should be noted that the new solution requires only the inversion of a small $J \times J$ matrix rather than a much larger $(H - K) \times (H - K)$ matrix and can be utilized to obtain solution of structures with sequential yielding or instabilities at the various degrees of freedom. The new solution is analogous to the modifications of the relative stiffnesses and loads caused by introducing a hinge in the Hardy-Cross moment distribution method.

A simple example is presented to illustrate the procedure and to demonstrate exact agreement with the correct solution. The new solution is analogous to the modifications of the relative stiffness and loads in the Hardy-Cross moment distribution method for a hinged support.

Figure 2 defines the geometry and degrees of freedom of an undamaged structure. The solution for an applied loads at 1, 2, and 3 is obtained from

$$\begin{bmatrix} \Delta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 48EI/L^3 & 0 & 0 \\ 0 & 8EI/L & 0 \\ 0 & 0 & 8EI/L \end{bmatrix}^{-1} \begin{bmatrix} P_1 \\ M_2 = 0 \\ M_3 = 0 \end{bmatrix}$$

which results in

$\delta(M,H) =$

	M \ H	1	2	3
	4	$-1/4$	0	$-3/4L$
	5	$L/8$	0	$1/4$
	6	$-1/4$	$-3/4L$	0
	7	$L/8$	$+1/4$	0
	8	$-1/4$	0	$3/4L$
	9	$-L/8$	0	$1/4$
K	10	$-1/4$	$3/4L$	0
K	11	$-L/8$	$+1/4$	0
		J	J	

The H degrees of freedom were not included as a subset of M to simplify the presentation. Correctness of the recommended procedure for the cut structure is established by demonstrating that the reaction are identical.

Let us cut the 10th and 11th degrees of freedom (see Fig. 1):

$\therefore B(J,K) =$

J \ K	10	11
1	1	0
2	$-L$	1

The released internal loads are

$$F_0(K) = f(K,H)P(H) = \begin{bmatrix} -1/4 \\ -L/8 \end{bmatrix} [P_1]$$

This results in [see Eq. (10)]

$$R(J) = [I(J,J) + B(J,K)f(K,J)]^{-1}B(J,K)[-F_0(K)] = \begin{bmatrix} 7 \\ -4L \end{bmatrix} [P_1/9]$$

The solution for the cut structure is therefore [see Eq. (11a)]

$$\bar{F}(M) = f(M,H)P(H) + f(M,J)R(J)$$

$$\begin{bmatrix} \bar{F}(4) \\ \bar{F}(5) \\ \bar{F}(6) \\ \bar{F}(7) \\ \bar{F}(8) \\ \bar{F}(9) \end{bmatrix} = \begin{bmatrix} -1/4 \\ L/8 \\ -1/4 \\ L/8 \\ -1/4 \\ -L/8 \end{bmatrix} [P_1] + \begin{bmatrix} -1/4 & 0 \\ L/8 & 0 \\ -1/4 & -3/4L \\ L/8 & 1/4 \\ -1/4 & 0 \\ -L/8 & 0 \end{bmatrix} \times \begin{bmatrix} 7 \\ -4L \end{bmatrix} [P_1/9] = \begin{bmatrix} -4 \\ 2L \\ -1 \\ L \\ -4 \\ -2L \end{bmatrix} [P_1/9]$$

and [see Eq. (12a)]

$$D(M) = \delta(M,H)P(H) + \delta(M,J)R(J)$$

$$\begin{bmatrix} \Delta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 48EI/L^3 & 0 \\ 0 & 8EI/L \end{bmatrix}^{-1} \begin{bmatrix} P_1 + 7P_1/9 \\ 0 - 4P_1L/9 \end{bmatrix} = \begin{bmatrix} P_1L^2/24EI \\ -P_1L^2/18EI \end{bmatrix}$$

The analysis of the cut structure as an original structure would result in

$$\begin{bmatrix} \Delta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 36EI/L^3 & 6EI/L^2 \\ +6EI/L^2 & 4EI/L \end{bmatrix}^{-1} \begin{bmatrix} P_1 \\ 0 \end{bmatrix}$$

with identical displacements and internal loads.

Equations (11, 12, and 14) represents a relatively simple procedure to obtain the structural behavior of a damaged structure. The equations can be applied with or without the assistance of digital computing machines and permit the analyst to visualize how the loads and deformations will change when the structure suffers a local failure, yielding, or instability. This can be very valuable in obtaining efficient designs since the designer can readily grasp how the loads will be redistributed when a local member is not adequate and thereby obtain a better proportioned structure. In addition, a more realistic ultimate strength capacity of the structure based upon limit rather than elastic analysis can be utilized.

Determination of the Wake behind a Bluff Body of Revolution at High Reynolds Numbers

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1. Introduction

The properties of the laminar wake formed by a bluff body in its steady motion through an incompressible fluid at high Reynolds number are not completely understood at present. Most of the existing theories have been primarily concerned with the two-dimensional case.[1-4] Although we are interested in the axisymmetric case, it is worthwhile to review the difficulties encountered in two dimensions.

The freestream solution due to Kirchhoff cannot describe a (complete) two-dimensional wake since it violates the intuitive result that the disturbance must die down at infinity downstream.[1] Batchelor[1] proposed a model with circulation in a finite wake. There are some experimental evidences[2] that, for a two-dimensional wake, the drag coefficient approaches a finite limit as the Reynolds number *Re* approaches infinity. Kirchhoff's model predicts a finite drag coefficient while Batchelor's model on the contrary leads to a zero drag coefficient as $Re \to \infty$. Recently Roshko[3] and Sychyev[4] have developed, independently, similar theories giving both a finite drag coefficient as $Re \to \infty$ and a finite wake for $Re \gg 1$. We shall see that the difficulty with Batchelor's model in two dimensions, which indeed makes it suspect, does not arise in the axisymmetric case. Batchelor has proven[5] that for a recirculating two-dimensional flow the vorticity ξ is constant in the cavity except possibly in thin boundary layers. The circulation in the cavity is induced by contact with the "outside," which could be for instance a rotating container or another flow (like in the case of a wake). If the outside tends to make the flow rotate in only one direction, no ambiguity arises about the response of the fluid in the cavity.† On the contrary, for a two-dimensional wake the flows on each side of the wake (the outside) tend to make the wake circulate in opposite direction. Consequently, the response of the wake is not obvious. Part of the wake may remain essentially stagnant (the near wake of Roshko and Sychyev), or the wake may be composed of two cells with circulation in opposite directions. Each of the two cells then separately obey Batchelor's theorem of constant vorticity (but with opposite signs in each cell).

For an axisymmetric flow experimental‡ and numerical[8] evidence indicates that no part of the fluid in the wake is stagnant. A superficial look at Fig. 1 might still give the impression that the outside flows on opposite sides of the wake work at cross purposes, hence raising doubts once more about the response of the fluid in the cavity. Actually the two apparent cells in Fig. 1 belong to the same continuous cell because of the axisymmetry of the flow. The vorticity does not change sign but is in the same azimuthal direction at each point. Then, if y is the distance from any point to the axis, Batchelor's theorem[5,7] indicates that ξ/y is uniform at each point in the wake (excepting thin boundary layers). In conclusion for the axisymmetric case the outside pulls consistently all around the wake and tends to make the fluid circulate in only one possible way; hence, there is no doubt that we have a completely circulating wake, obeying Batchelor's theorem.[5,7]

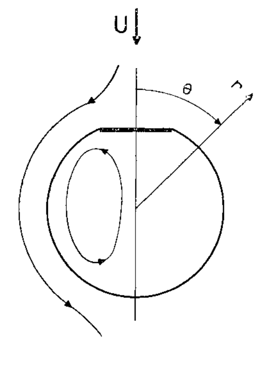

Fig. 1 Sketch of wake and streamlines.

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† One has only to think about the fluid in a circular cylinder rotating at uniform angular velocity. Once a steady state is reached, the fluid rotates at the same angular velocity (uniform vorticity).

‡ For instance in the case of spherical cap bubbles,[6] the circulation in the wake can be visualized very easily with small air bubbles. For the related problem of water bells[7] (the water sheets are the outside) the circulation is clearly seen by injection of smoke. In both cases the assumption that Batchelor's theorem for axisymmetric flow[5] holds in the whole cavity leads to results[6,7] well checked experimentally.